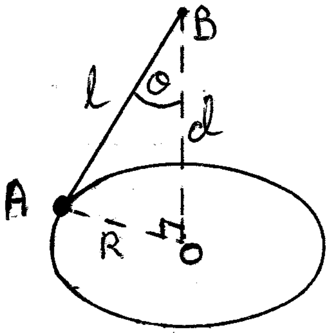
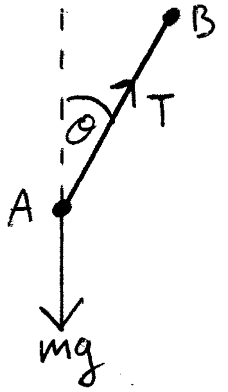
Mechanics Examples Sheet 9 – Solutions

1.





Aside: As A rotates, the string AB traces out the surface of a cone. Hence, the system is known as a <u>conical pendulum</u>.

For the particle to be in vertical equilibrium, we must have

$$T\cos\theta = mg.$$

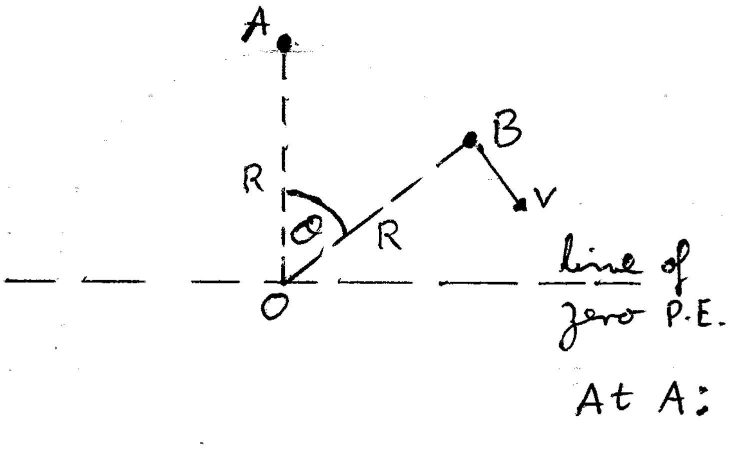
As the particle is performing uniform circular motion a force of constant magnitude acts towards the origin O. This force is the product of the mass and the radial acceleration: $mR\dot{\theta}^2$ or equivalently, mv^2/R. Thence, ~~for horizontal equilibrium~~

$$T\sin\theta = \frac{mv^2}{R} = \frac{mv^2}{l\sin\theta}.$$

On eliminating T, we obtain

$$v^2 = \frac{gl\sin^2\theta}{\cos\theta}$$

2.



Initially the particle is stationary at A. When the particle is at B the radius makes an angle θ with the vertical.

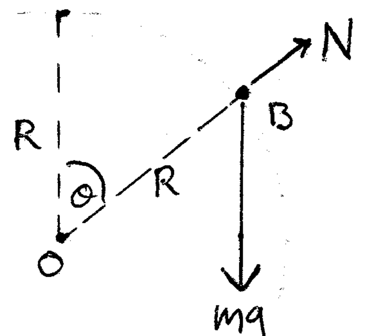
At A: P.E. $= mgR$, K.E. $= 0$

At B: P.E $= mgR\cos\theta$, K.E $= \frac{1}{2}mv^2$

Employing conservation of energy, we have

$$mgR = mgR\cos\theta + \frac{1}{2}mv^2$$

$$\Rightarrow \quad \underline{\underline{V^2 = 2gR(1-\cos\theta)}}$$



At B, let the normal reaction of the sphere on the particle be N.

The radial acceleration acts towards the origin O. Hence,

$$\frac{mV^2}{R} = \text{Net radial force towards O}$$

$$= mg\cos\theta - N$$

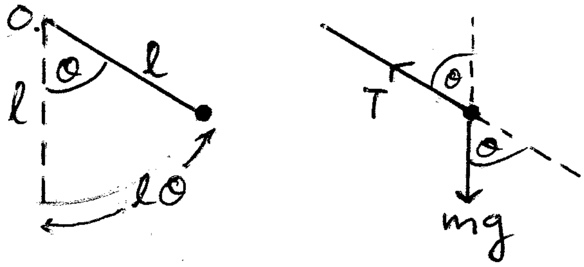
$$2mg(1-\cos\theta) = mg\cos\theta - N$$

$$\Rightarrow \underline{\underline{N = mg(3\cos\theta - 2)}}$$

When the particle leaves the surface of the sphere the normal reaction vanishes. Thus,

$$N = 0 \quad \Rightarrow \quad \underline{\underline{\cos\theta = \frac{2}{3}}}$$

3.



Radial component towards O:

$$-T + mg\cos\theta = -ml\dot{\theta}^2$$

Transverse component:

$$-mg\sin\theta = ml\ddot{\theta}$$

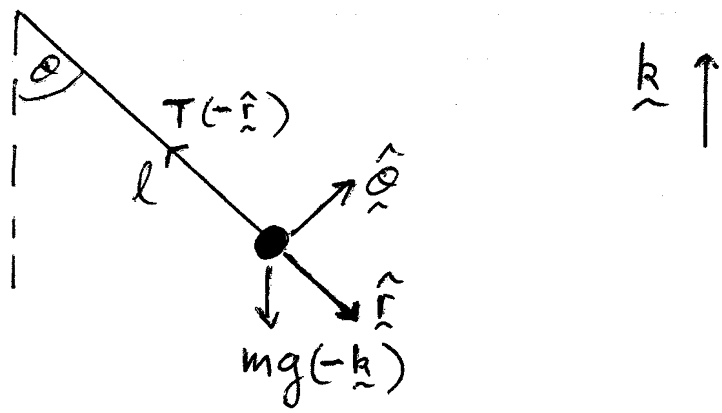
$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

equation of SHM

$$\Rightarrow \underline{\underline{\ddot{\theta} + \omega^2\theta = 0}}, \quad \sin\theta \simeq \theta \text{ for small } \theta,$$

where $\omega = \sqrt{\frac{g}{l}}$ $\therefore$ period, $\tau = \frac{2\pi}{\omega} = \underline{\underline{2\pi\sqrt{\frac{l}{g}}}}$

3. (Alternative)



The acceleration of the particle is

$$\mathbf{a} = -l\dot{\theta}^2 \hat{\mathbf{r}} + l\ddot{\theta}\hat{\boldsymbol{\theta}} .$$

The gravitational force acting on the particle is

$$-mg\mathbf{k} = -mg[\cos\theta(-\hat{\mathbf{r}}) + \sin\theta(\hat{\boldsymbol{\theta}})].$$

The tension in the string is

$$\mathbf{T} = T(-\hat{\mathbf{r}}).$$

Therefore, the equation of motion is

$$\mathbf{F} = m\mathbf{a} = -mg\mathbf{k} + \mathbf{T}$$

$$\Rightarrow m(-l\dot{\theta}^2\hat{\mathbf{r}} + l\ddot{\theta}\hat{\boldsymbol{\theta}}) = mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) - T\hat{\mathbf{r}}$$

On comparing components of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$:

$$-ml\dot{\theta}^2 = mg\cos\theta - T$$

and

$$ml\ddot{\theta} = -mg\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \omega^2\theta = 0 \quad , \text{ for small } \theta,$$

where $\omega = \sqrt{\frac{g}{l}}$.

Period, $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$